CAT 8

Discrete Morse Theory

Note 0

a) The n-simplex is ACYCLIC, i.e., has the homology of a point. But if we want to actually COMPUTE its Homology groups, we must build a simplicial complex with $\underline{2^n - 1}$ simplices !! Can we do better ?

b) Our goal today is to reduce a LARGE chain complex

$$\boxed{\cdots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0}$$

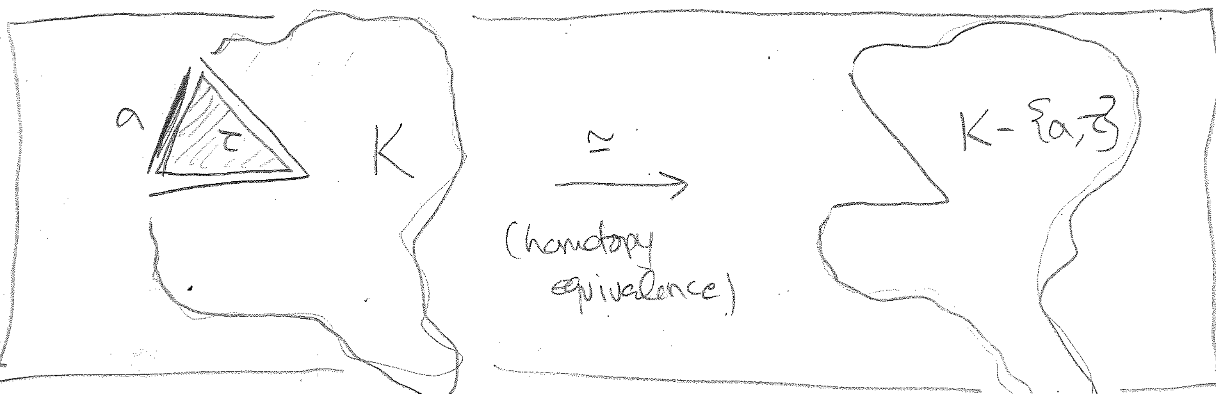
coming from some simplicial complex K to a MUCH SMALLER chain complex

$$\boxed{\cdots \longrightarrow M_2 \xrightarrow{e_2} M_1 \xrightarrow{e_1} M_0 \longrightarrow 0}$$

with the SAME HOMOLOGY, i.e.,

$$\boxed{H_i(C_\bullet, d_\bullet) \simeq H_i(M_\bullet, e_\bullet)}$$

c) The key observation in all this is the fact that a pair of simplices $\alpha < \tau$ in K can be REMOVED without changing homology provided that τ is the only simplex in the set $\{\gamma \in K \mid \gamma \geq \alpha \text{ and } \dim\gamma - \dim\alpha = 1\}$



d) The basic idea is to remove simplices, two-at-a-time, as many times as possible, without changing the homology groups.

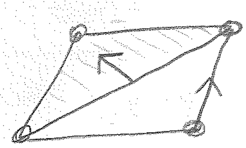
Def 1

a) Let K be a simplicial complex. A <u>PARTIAL MATCHING</u> on K is a collection $\Sigma = \{(\alpha_0 < \tau_0)\}$ of simplex-pairs satisfying

a) if $(\alpha < \tau) \in \Sigma$ then $\dim \tau = \dim \alpha + 1$.

b) if $(\alpha < \tau) \in \Sigma$ then no other pair in Σ contains either α or τ.

Such matchings are usually illustrated via arrows from α to τ, eg,



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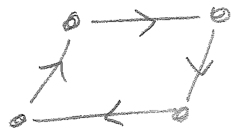
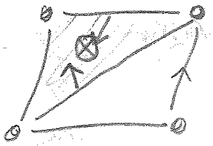


The following are <u>NOT</u> partial matchings:



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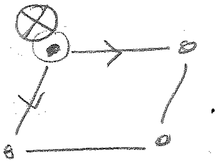

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b) The simplices of K which do <u>NOT</u> appear in Σ are called <u>Σ - CRITICAL</u>, or usually just CRITICAL.

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c) A Σ - <u>PATH</u> is a sequence of paired simplices that fits together like this:

$$\boxed{\underbrace{\alpha_0 < \tau_0}_{\text{in } \Sigma} > \underbrace{\alpha_1 < \tau_1}_{\text{in } \Sigma} > \underbrace{\alpha_2 < \tau_2}_{\text{in } \Sigma} > \cdots > \underbrace{\alpha_k < \tau_k}_{\text{in } \Sigma}}$$

where all α_i's have the same dimension.

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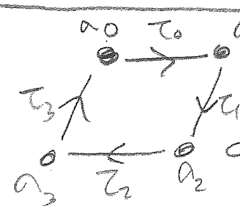
d) We call Σ an <u>ACYCLIC</u> partial matching if none of its paths are cycles, i.e., $\nexists$ path

$$\alpha_0 < \tau_0 > \cdots > \alpha_k < \tau_k$$

with $k > 0$ and $\alpha_0 < \tau_k$



NOT acyclic!

Idea 2

R. Forman, "Morse Theory for cell complexes"

If we are given an acyclic partial matching Σ on K, then the homology of K can be computed by a "much smaller" chain complex

$$\boxed{\cdots \xrightarrow{e_3} M_2 \xrightarrow{e_2} M_1 \xrightarrow{e_1} M_0 \longrightarrow 0}$$

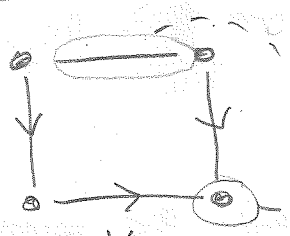
called the "Morse complex" of Σ.

where the chain groups are

$$\boxed{M_i = \bigoplus_{\substack{\dim \sigma = i \\ \sigma \text{ is } \Sigma\text{-critical}}} \mathbb{F}},$$

and the boundary operator $e_\bullet$ comes from "chasing gradient paths" between critical cells.

Eg 3

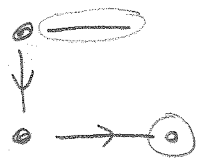


K

There are only two critical cells, so the Homology here can be computed by

$$0 \longrightarrow \mathbb{F} \xrightarrow{e_1} \mathbb{F} \longrightarrow 0$$

We "know" in this case that e_1 must be zero, since K is (homotopy-equivalent to) a circle. There are two Σ-paths from the 1-dim critical cell to the 0-dim critical cell, i.e.,



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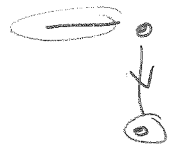

These each contribute ± 1 to e_1, and in this case those contributions will cancel, so $e_1 \equiv 0 : \mathbb{F} \to \mathbb{F}$

Q4 How do we (efficiently) determine the mysterious Morse boundary operator $e_\bullet$? What is the formula that collects all those Σ-path contributions?

Ans 5

a) Let's look at the effect of "removing a single pair" $(\alpha < \tau)$ in Σ on the boundary matrix, eg

$$d_i = \begin{array}{c|c} & \begin{array}{cccc} (\tau) & (\gamma) & \cdots & \leftarrow \text{(i-simplices)} \end{array} \\ \begin{array}{c} (\alpha) \\ (\delta) \\ \text{(i-1)-simplices} \end{array} & \begin{bmatrix} \boxed{c_{\tau,\alpha}} & \cdots & \\ \vdots & c_{\gamma,\delta} & \cdots \\ & \vdots & \end{bmatrix} \end{array}$$

Note: $c_{\tau,\alpha} = \pm 1$ because $\alpha < \tau$

This is $\langle \partial \tau, \alpha \rangle$

We would like to perform row and column ops so that $c_{\tau,\alpha}$ is the ONLY nonzero entry in its row and column. How does this impact some generic entry $c_{\gamma,\delta}$?

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b) Well, to clear out $c_{\gamma,\alpha}$ we need the COLUMN OPERATION

$$\boxed{\mathrm{Col}(\gamma) \longleftarrow \mathrm{Col}(\gamma) - \left[\frac{c_{\gamma,\alpha}}{c_{\tau,\alpha}}\right] \cdot \mathrm{Col}(\tau)}$$

And to clear out $c_{\tau,\delta}$ we need a ROW OPERATION

$$\boxed{\mathrm{Row}(\delta) \longleftarrow \mathrm{Row}(\delta) - \left[\frac{c_{\tau,\delta}}{c_{\tau,\alpha}}\right] \cdot \mathrm{Row}(\alpha)}$$

In EITHER case, the modified entry in $\mathrm{Col}(\gamma), \mathrm{Row}(\delta)$ is

$$\boxed{c^{\text{new}}_{\gamma,\delta} = c^{\text{old}}_{\gamma,\delta} - \left[\frac{c_{\tau,\delta} \cdot c_{\gamma,\alpha}}{c_{\tau,\delta}}\right]}$$

Def 6 Here, we had $\underline{\gamma > \alpha < \tau > \delta}$, so we now know the contribution of a single Σ-path: the <u>MULTIPLICITY</u> of a path $\underline{p} = (\alpha_0 < \tau_0 > \alpha_1 < \tau_1 > \cdots > \alpha_k < \tau_k)$ is the product

$$\boxed{m(\underline{p}) = \left(\frac{-1}{c_{\tau_0,\alpha_0}}\right) \cdot (c_{\tau_0,\alpha_1}) \cdot \left(\frac{-1}{c_{\tau_1,\alpha_1}}\right) \cdots \left(\frac{-1}{c_{\tau_k,\alpha_k}}\right)}$$

This is just a NUMBER, i.e., an element of $\mathbb{F}$.

Thm 7 Let K be a finite simplicial complex and Σ an acyclic partial matching on K. Then, the homology $H_\bullet(K;\mathbb{F})$ can be computed by the chain complex

$$\boxed{\cdots \xrightarrow{e_3} M_2 \xrightarrow{e_2} M_1 \xrightarrow{e_1} M_0 \longrightarrow 0}$$

where $M_i = \bigoplus\limits_{\substack{\dim \alpha = i \\ \alpha \text{ critical}}} \mathbb{F}$, and the boundary matrix $e_i : M_i \longrightarrow M_{i-1}$ is given as follows.

$$e_i = \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} & (\gamma) \\ (\delta) & \left[\begin{array}{ccc} & \vdots & \\ \cdots & \boxed{e_{\gamma,\delta}} & \cdots \\ & \vdots & \end{array}\right] \end{array}$$

(γ): critical, dimension i

(δ): critical, dimension $i-1$

$$\boxed{e_{\gamma,\delta} = c_{\gamma,\delta} + \sum_{\substack{\rho = (\alpha_0 < \cdots < \tau_k) \\ \Sigma\text{-path}}} c_{\gamma,\alpha_0} \cdot m(\rho) \cdot c_{\tau_k,\delta}}$$

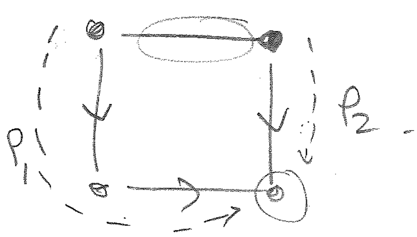
The summand is what you get by computing the multiplicity of ρ "augmented" by critical cells, i.e.,

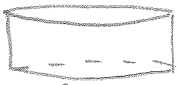
$$\underbrace{\gamma}_{\text{critical}} > \overbrace{\boxed{\underbrace{\alpha_0 < \tau_0}_{\text{in } \Sigma} > \underbrace{\alpha_1 < \tau_1}_{\text{in } \Sigma} > \cdots > \underbrace{\alpha_k < \tau_k}_{\text{in } \Sigma}}}^{\rho} > \underbrace{\delta}_{\text{critical}}$$

Note 8 When working with $\mathbb{F} = \mathbb{Z}/2$, the matrix entry $e_{\gamma,\delta} \in \{0,1\}$ just counts the parity (even $=0$, odd $=1$) of the number of Σ-paths that "flow from γ to δ".

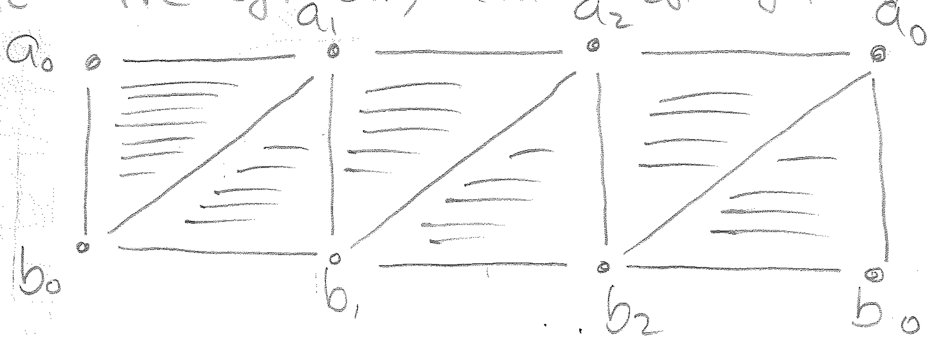
This example had two paths, so

$$\cdots \longrightarrow M_1 \xrightarrow{0} M_0 \longrightarrow 0 \quad \checkmark$$



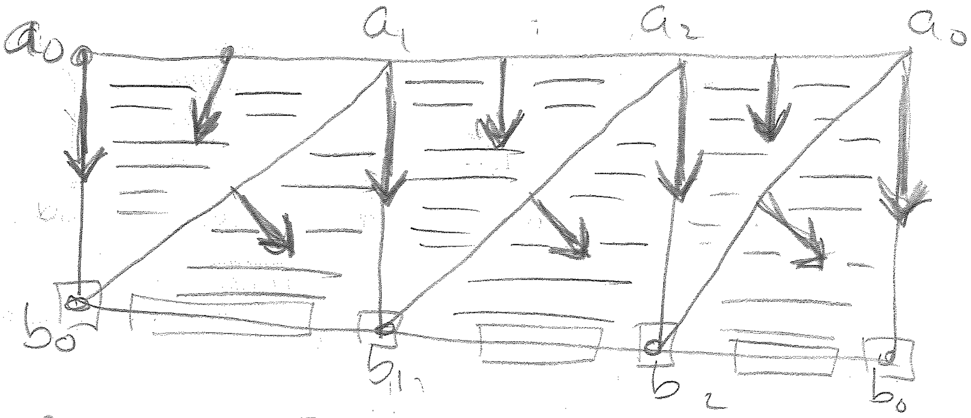
Eg 9 Show that a cylinder  has the same homology as the base circle

a) Triangulate the cylinder, (note left-right edges are same)

K = 

Note:– Left side is identified with right side

To show that this has the same homology as the circle, just build an acyclic partial matching whose critical set is an embedded circle in K:

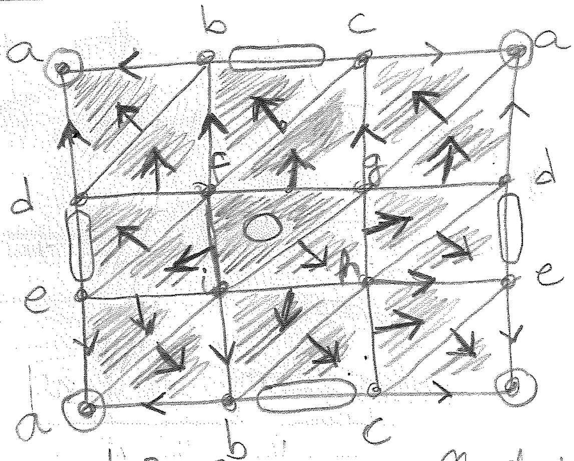


Critical →

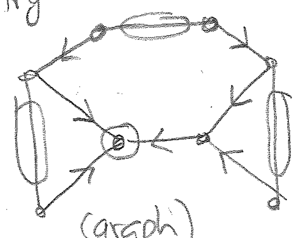
There are no Σ_T paths flowing between the critical cells, so the Morse boundary is just the restriction of the usual simplicial boundary to the b_0 - simplices

+ Bonus: The chain homotopy is given by arrows, eg $\Theta(a_0a_1) = a_0a_1b_0 + a_1b_0b_1$, and $\Theta(a_1) = a_1b_1$ etc

b) Torus



or, try:



(graph)

$0 \to \mathbb{F}^3 \xrightarrow{[0\,0\,0]} \mathbb{F} \to 0$

Counting Σ-paths mod 2 gives a small chain complex

$0 \to (\mathbb{Z}/2) \xrightarrow{\begin{bmatrix}0\\0\end{bmatrix}} (\mathbb{Z}/2)^2 \xrightarrow{[0\,0]} (\mathbb{Z}/2) \to 0$

[fgi] [bc],[de] [a]

easy !!

MUTATIONS

Discrete Morse Theory adapts nicely to other (co) homology computations — PERSISTENT Homology and SHEAF cohomology.

I. PERSISTENCE : Here are two filtrations of a

$\boxed{\alpha}$ 1-simplex (numbers on simplices indicate which stage they are born in):

Fails ⊗ 

✓ Works 

When can we "pair and remove" two simplices without changing the persistent homology?

$\boxed{\beta}$ Every filtration $K_0 \subseteq K_1 \subseteq \dots \subseteq K_n = K$ of a simplicial complex K is "equivalent to" a single function

"birth".

$$\underline{b} : K \to \mathbb{N},$$

where $\underline{b}(\sigma) = \min\{i \in \{0,\dots,n\} \mid \sigma \in K_i\}$,

so $\sigma \leq \tau$ means $\underline{b}(\sigma) \leq \underline{b}(\tau)$. $\longrightarrow (*)$

Def $\boxed{\gamma}$ An acyclic partial matching Σ on K is COMPATIBLE with the filtration $K_0 \subseteq \dots \subseteq K_n = K$ if it only pairs simplices with the SAME $\underline{b}$-value, i.e.,

$$(\sigma < \tau) \in \Sigma \Rightarrow \underline{b}(\sigma) = \underline{b}(\tau)$$

[In other words, restricting Σ to K_i produces a genuine partial matching on K_i].

THM $\boxed{\delta}$ [Mischaikow-Nanda, "Morse Theory for filtrations"]

If Σ is compatible with a filtration $K_0 \subseteq \dots \subseteq K_n = K$ of K, then the Morse chain complexes include; i.e., letting $M^i_\bullet$ be the Morse complex associated to the restriction of Σ to K_i, we get injective chain maps

$$\boxed{M^0_\bullet \xhookrightarrow{\iota_0} M^1_\bullet \xhookrightarrow{\iota_1} M^2_\bullet \xhookrightarrow{\iota_2} \cdots \hookrightarrow M^n_\bullet} \leftarrow \text{"filtered Morse complex"}$$

(i.e., all such squares commute:

$$\begin{tikzcd} & \arrow[d] & \arrow[d] & \\ \cdots \arrow[r, hook] & M^i_j \arrow[r, hook, "\iota_j"] \arrow[d, "e^i_j"] & M^{i+1}_j \arrow[r] \arrow[d, "e^{i+1}_j"] & \cdots \\ \cdots \arrow[r, hook] & M^i_{j-1} \arrow[r, hook, "\iota_{j-1}"'] \arrow[d] & M^{i+1}_{j-1} \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd}$$

Here, M^i_j = chain group generated by Σ-critical simplices with $\underline{b}$-values $\leq i$ and dimension $= j$, and e^i_j is the Morse boundary operator)

Moreover, the Persistent Homology groups of the filtered Morse complex $\underline{\text{coincide}}$ with those of K.
"are isomorphic"

"$\underline{\text{Pf}}$" If Σ is filtration-compatible, then any Σ-path

$$\rho = (\alpha_0 < \tau_0 > \alpha_1 < \tau_1 > \cdots > \alpha_n < \tau_n)$$

is $\underline{b}$-nonincreasing! Every time we see a "<", the $\underline{b}$ value is the same across it. And if we see a ">", then $\underline{b}$ can not increase by $(*)$ (see $\boxed{B}$ on previous page). Thus, the boundary map e^{i+1}_j precisely equals e^i_j when restricted to the smaller Morse complex $M^i_j \subseteq M^{i+1}_j$.

II SHEAVES: What if we had a sheaf $\mathcal{F}$ on K? Could we compute sheaf cohomology $H^\bullet(K;\mathcal{F})$ by using discrete Morse Theory?

α The main difference is that the co-boundary map $d^\bullet_{\mathcal{F}} : C^\bullet(K;\mathcal{F}) \to C^{\bullet+1}(K;\mathcal{F})$ has the block-form

$$\boxed{d^\bullet_{\mathcal{F}} = \begin{array}{c|c} & \sigma \\ \tau & \begin{bmatrix} & \vdots & \\ \cdots & \boxed{\star_{\sigma,\tau}} & \cdots \\ & \vdots & \end{bmatrix} \end{array}}$$

Where $\boxed{\star_{\sigma,\tau}} = \underbrace{\langle \partial\tau, \sigma\rangle}_{\text{number } \{\pm 1, 0\}} \cdot \mathcal{F}(\sigma \le \tau)$

Matrix $\mathcal{F}(\sigma) \to \mathcal{F}(\tau)$, Restriction map

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β If we have a pair $(\sigma < \tau)$, then we need $\boxed{\star}$ to be $\boxed{\text{INVERTIBLE}}$ to define the necessary row and column operations! [See Ans 5 above]; and in this case, the multiplicity of a Σ-path <u>with coefficients in $\mathcal{F}$</u> is just given as follows:

For $\rho = (\sigma_0 < \tau_0 > \sigma_1 < \tau_1 > \cdots > \sigma_k < \tau_k)$,

$$m_{\mathcal{F}}(\rho) = \boxed{-\star^{-1}_{\sigma_0,\tau_0}} \circ \boxed{\star_{\sigma_1,\tau_0}} \circ \boxed{-\star^{-1}_{\sigma_1,\tau_1}} \circ \cdots \circ \left[-\star^{-1}_{\sigma_k \tau_k}\right]$$

This is a LINEAR MAP $\mathcal{F}(\tau_k) \longrightarrow \mathcal{F}(\sigma_0)$

γ So, Morse boundary map $e^\bullet_{\mathcal{F}}$ has the following "block" relating critical simplex $\underset{\dim d}{\gamma}$ to critical simplex $\underset{\dim d+1}{\delta}$:

$$\boxed{e^\bullet_{\mathcal{F}}\Big|_{\gamma,\delta} = \star_{\gamma,\delta} + \sum_{\substack{\rho \text{ a } \Sigma\text{-path} \\ = (\sigma_0 < \tau_0 > \cdots > \sigma_k < \tau_k)}} \boxed{\star_{\sigma_0,\delta}} \circ m_{\mathcal{F}}(\rho) \circ \boxed{\star_{\gamma,\tau_k}}}$$

This is a linear map $\mathcal{F}(\gamma) \longrightarrow \mathcal{F}(\delta)$ as desired.

$\boxed{\delta}$ **Thm** (Sköldberg)

Let $\mathcal{F}$ be a sheaf on a simplicial complex K and let Σ be an $\mathcal{F}$-compatible acyclic partial matching on K, i.e.,

$$(\sigma < \tau) \in \Sigma \Rightarrow \mathcal{F}(\sigma < \tau) \text{ is invertible.}$$

Then, there is a cochain complex

$$\boxed{M_{\mathcal{F}}^{0} \xrightarrow{e_{\mathcal{F}}^{0}} M_{\mathcal{F}}^{1} \xrightarrow{e_{\mathcal{F}}^{1}} \cdots}$$

Where $M_{\mathcal{F}}^{i} = \bigoplus_{\substack{\sigma \text{ critical,} \\ \dim \sigma = i}} \mathcal{F}(\sigma)$, and $e_{\mathcal{F}}^{\bullet}$ is given by the formula in $\boxed{\gamma}$, so that $H^{\bullet}(K, \mathcal{F})$ is exactly the cohomology of $M_{\mathcal{F}}^{\bullet}$!!

FURTHER READING ON D.M.T

There are two papers on using discrete Morse theory for (A) persistence and (B) sheaves:

(A) "Morse theory for filtrations & efficient computation of persistent homology" – Discrete & Comput. geometry, 2013

(B) "Discrete Morse Theory for computing cellular sheaf cohomology" – Foundations of Comput. maths, 2016.

Here you can find detailed proofs of all the results. The basic idea is this: show that if you remove ONE Σ-pair $(x > y)$, then we get a chain homotopy equivalence. Then reiterate, removing another pair, etc.